Math 4210 Assignment 3

$\Rightarrow$ change Call to Put or vice versa.

Due time: April 17th, 2024, 23:59

Question 1

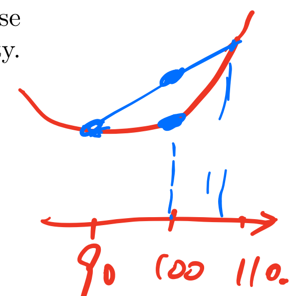
We observe the prices (at time $t = 0$) of the following European call/put options on the market. Suppose that the interest rate $r = 0$, and the initial price of the underlying stock is $S_0 = 100$. Please construct a portfolio, using these options together with the cash (bank account), to find an arbitrage opportunity.

① Put-call parity.

② Say we have 3 options with same type: use the convexity regarding strikes

③ If $r=0$, the $P(T=T_1) \leq P(T=T_2)$, $T_1 \leq T_2$

Convex.



Option Type	Strike	Maturity	Option Price at time $t = 0$
Put	90	2	6
Call	100	1	11
Put	110	2	14

Question 2

Lily has a 15 years home mortgage. She needs to pay \$12000 at the end of each quarter (i.e. every 3 months) for the next 15 years. The interest on the loan is compounded quarterly with annual nominal rate 2%.

She would like to refinance the loan to a 30 year loan which is paid monthly, with annual nominal rate 3% compounded monthly. What is her new monthly payment ?

Notice: For both loans, the payments are made at the end of each period, i.e. there is no payment at the initial time 0.

Plan 1 and Plan 2

PV of plan 1 = PV of plan 2.

$$PV_1 = \sum_{i=1}^{4\times 15} \frac{12000}{(1+\frac{2\%}{4})^i} = PV_2 = \sum_{i=1}^{30\times 12} \frac{\text{Cash flow}}{(1+\frac{3\%}{12})^i}$$

$\Rightarrow$ Cash flow ≈ 1441.42

Question 3

1. Apply Itô formula on $f(t, x) = tx$ to prove that
$$tB_t = \int_0^t s\, dB_s + \int_0^t B_s\, ds$$
2. Deduce from above result that
$$\int_0^t B_s\, ds = \int_0^t (t - s)\, dB_s.$$
3. Compute
$$\mathbb{E}\Big[\int_0^t B_s\, ds\Big] \quad \text{and} \quad \mathrm{Var}\Big[\int_0^t B_s\, ds\Big].$$

① Itô's formula.

② Itô's Isometry. Expectation of Itô's integral.

Q3. 1. $f(t,x) = tx$. $f \in C^{1,2}([0,T]\times\mathbb{R})$

Then $df(t,B_t) = d(tB_t)$

$= dt \cdot B_t + t \cdot dB$ (product rule)

Then we integrate: $\underset{\underset{tB_t}{\shortparallel}}{f(t,B_t)} - f(0,B_0) = \int_0^t B_s ds + \int_0^t s\, dB_s$.

$\Rightarrow tB_t = \int_0^t B_s ds + \int_0^t s\, dB_s$

2. It suffices to notice that $tB_t = t \cdot \int_0^t 1\, dB_s$

So we have: $t \cdot \int_0^t dB_s = \underline{\int_0^t B_s ds} + \int_0^t s\, dB_s$

$\Rightarrow \int_0^t B_s ds = \int_0^t t\, dB_s - \int_0^t s\, dB_s$

$= \int_0^t (t-s)\, dB_s$

3. We need to compute $E[\underline{\int_0^t B_s ds}]$, $Var[\int_0^t B_s ds]$

Since $\int_0^t B_s ds = \underbrace{\int_0^t (t-s) dB_s}_{\text{Ito integral}}$

So $E[\int_0^t (t-s) dB_s] = 0$, $E[(\int_0^t (t-s) dB_s)^2]$

$= E[\int_0^t (t-s)^2 \cdot ds]$ (Ito's Isometry)

$= \int_0^t (t-s)^2 ds.$

$= \int_0^t (t^2 - 2st + s^2) ds$

$$= t^3/3.$$

$$\Rightarrow \mathrm{Var}\left(\int_0^t B_s ds\right) = E\left[\left(\int_0^t B_s ds\right)^2\right] - E\left[\int_0^t B_s ds\right]^2$$
$$= t^3/3$$

Question 4

We consider a continuous time market, where the interest rate $r \geq 0$, and the risky asset $S = (S_t)_{0 \leq t \leq T}$ follows the Black-Scholes model with initial value $S_0 = 1$, drift μ and volatility $\sigma > 0$ (without any dividend), so that

$$S_t = S_0 \exp\left((\mu - \sigma^2/2)t + \sigma B_t\right).$$

① self-financing portfolio
② Itô's formula.
③ Black-Scholes ($\mathbb{P}$, $\mathbb{Q}$)
④ MGF
⑤ PDE

$\mathbb{Q}$ is also called equivalent martingale measure.

$\mathbb{P}$ is equivalent to $\mathbb{Q}$

Under $\mathbb{Q}$, discounted financial derivatives are martingales

$\tilde{S}_t$ is $\mathbb{Q}$-martingale

$\tilde{\Pi}_t$ . . . .

$e^{-rT} g(S_T)$ . . . —

– If we have martingale M_t.
then dM_t has no drift term.

a) A self-financing portfolio is given by (x, ϕ), where x represents the initial wealth of the portfolio, and ϕ_t represents the number of risky asset in the portfolio at time t. Let $(\Pi_t^{x,\phi})_{t \in [0,T]}$ be the wealth process of the portfolio, write down the dynamic of $\Pi^{x,\phi}$ in form of

$$d\Pi_t^{x,\phi} = \alpha_t dt + \beta dB_t, \quad \text{for some (to be founded) process } (\alpha, \beta).$$

b) There exists a unique risky-neutral probability $\mathbb{Q}$, together with a Brownian motion $B^{\mathbb{Q}}$ under the probability measure $\mathbb{Q}$.

Please give the expression of the process S_t as a function of $(t, B_t^{\mathbb{Q}})$.

c) We first consider an option with payoff $g(S_T) = S_T^2$ at maturity T.

– Compute the value

$$V_0 := \mathbb{E}^{\mathbb{Q}}\left[e^{-rT} S_T^2\right].$$

① $v(t,x) = E^{\mathbb{Q}}[e^{-r(T-t)} S_T^2 \mid S_t = x]$

– Let $v(t, x) := x^2 \exp\left((r + \sigma^2)(T - t)\right)$, compute $\partial_t v$, $\partial_x v$ and $\partial_{xx}^2 v$, and then check that v satisfies the equation

$$\partial_t v(t,x) + rx\partial_x v(t,x) + \frac{1}{2}\sigma^2 x^2 \partial_{xx}^2 v(t,x) - rv(t,x) = 0, \qquad v(T,x) = x^2.$$

– Let $\tilde{S}_t := e^{-rt} S_t$ and $\tilde{g}(S_t) := e^{-rt} g(S_t)$ for all $t \geq 0$. Notice that both S_t and $\tilde{S}_t$ are functions of (t, B_t), apply the Itô formula on $e^{-rt} v(t, S_t)$ to deduce that

$$\tilde{g}(S_T) = V_0 + \int_0^T \phi_t d\tilde{S}_t, \quad \text{where } \phi_t := \partial_x v(t, S_t).$$

Deduce that V_0 is the (no-arbitrage) price of the option $g(S_T) = S_T^2$.

(a): $\Pi_t^{x,\phi}$ is a self-financing portfolio $K_2 - S_T$ $\mathbb{1}_{K_1 < S_T \leq K_2}$

$$d\Pi_t = (\Pi_t - \phi_t S_t) r dt + \phi_t dS_t$$
$$= (\Pi_t - \phi_t S_t) r dt + \phi_t(\mu S_t dt + \sigma S_t dB_t)$$
$$= ((\Pi_t - \phi_t S_t) r + \phi_t \mu S_t) dt + \sigma \phi_t S_t dB_t$$

(b). Under probability measure $\mathbb{Q}$, $S_t = S_0 \exp((r - \sigma^2/2)t + \sigma B_t^{\mathbb{Q}})$

(c). 1.

$$V_0 = \mathbb{E}^{\mathbb{Q}}[e^{-rT}S_T^2]$$ (just the discounted payoff under $\mathbb{Q}$)

$$= e^{-rT}\mathbb{E}^{\mathbb{Q}}[S_0^2 \exp(2(r-\sigma^2/2)\cdot T + 2\sigma B_T^{\mathbb{Q}}]$$

$$= e^{-rT+2rT-\sigma^2 T}\,\mathbb{E}^{\mathbb{Q}}[e^{2\sigma B_T^{\mathbb{Q}}}]$$

(Moment-Generating function)

$$= e^{rT-\sigma^2 T}\cdot e^{\frac{1}{2}\cdot(2\sigma)^2\cdot T}$$

$$\boxed{V_0 = e^{rT+\sigma^2 T}}$$

$V(0,S_0) = S_0^2\exp((r+\sigma^2)T) = V_0$

– Let $v(t,x) := x^2\exp\left((r+\sigma^2)(T-t)\right)$, compute $\partial_t v$, $\partial_x v$ and $\partial_{xx}^2 v$, and then check that v satisfies the equation

$$\partial_t v(t,x) + rx\partial_x v(t,x) + \frac{1}{2}\sigma^2 x^2 \partial_{xx}^2 v(t,x) - rv(t,x) = 0, \qquad v(T,x) = x^2.$$

$$\partial_t V(t,x) = -(r+\sigma^2)\,x^2\exp((r+\sigma^2)(T-t))$$

$$= -(r+\sigma^2)\,V(t,x)$$

$$\partial_x V(t,x) = 2x\exp((r+\sigma^2)(T-t))$$

$$= \frac{2}{x}\,V(t,x)$$

$$\partial_{xx}^2 V(t,x) = 2\exp((r+\sigma^2)(T-t))$$

$$= \frac{2}{x^2}\,V(t,x)$$

$$\partial_t V(t,x) + rx\,\partial_x V(t,x) + \frac{1}{2}\sigma^2 x^2 \partial_{xx}^2 V(t,x)$$

$$= \left(-(\cancel{r}+\cancel{\sigma^2}) + r\cdot\cancel{x}\cdot\frac{2}{\cancel{x}} + \frac{1}{\cancel{2}}\cdot\cancel{\sigma^2}\cdot\cancel{x^2}\cdot\frac{\cancel{2}}{\cancel{x^2}}\right)V(t,x)$$

$$= +r\,V(t,x)$$

3. – Let $\tilde{S}_t := e^{-rt}S_t$ and $\tilde{g}(S_t) := e^{-rt}g(S_t)$ for all $t \geq 0$. Notice that both S_t and $\tilde{S}_t$ are functions of (t, B_t), apply the Itô formula on $e^{-rt}v(t, S_t)$ to deduce that

$$\tilde{g}(S_T) = V_0 + \int_0^T \phi_t d\tilde{S}_t, \quad \text{where } \phi_t := \partial_x v(t, S_t).$$

Deduce that V_0 is the (no-arbitrage) price of the option $g(S_T) = S_T^2$.

$$d(e^{-rt}V(t,S_t)) = -re^{-rt}V(t,S_t)dt + e^{-rt}dV(t,S_t)$$

$$dV(t,S_t) = \partial_t V(t,S_t)dt + \partial_x V(t,S_t)dS_t + \frac{1}{2}\partial_{xx}^2 V(t,S_t)d[S]_t$$

$$= \partial_t V(t,S_t)dt + \partial_x V(t,S_t)dS_t + \frac{1}{2}\sigma^2 S_t^2 V(t,S_t)dt$$

Then. $d(e^{-rt}V(t,S_t)) = -re^{-rt}V(t,S_t)dt$ $\quad dS_t = rS_t dt + \sigma S_t dB_t$

$$+ e^{-rt}\left(\partial_t V(\ )dt + \partial_x V(\ )dS_t + \frac{1}{2}\sigma^2 S_t^2 V(\ )dt\right)$$

$$= e^{-rt}\underbrace{\left(-rV + \partial_t V + \partial_x V \cdot rS_t + \frac{1}{2}\sigma^2 S_t^2 V\right)}_{=0 \text{ by the PDE}}dt$$

$$+ e^{-rt}\partial_x V \cdot \sigma S_t dB_t$$

$$= e^{-rt}\partial_x V \cdot \sigma S_t dB_t$$

Notice that $d\tilde{S}_t = d(e^{-rt}S_t) = e^{-rt}\sigma S_t dB_t$

Thus: $d(e^{-rt}V(t,S_t)) = 2 \cdot S_t \exp((r+\sigma^2)(T-t))d\tilde{S}_t$

$$\Rightarrow e^{-rT}V(T,S_T) - V(0,S_0) = \int_0^T \partial_x V(t,S_t)d\tilde{S}_t$$

$$\Rightarrow \tilde{g}(S_T) = \boxed{V_0} + \int_0^T \partial_x V(t,S_t)d\tilde{S}_t$$

Π is self financing, $d\tilde{\Pi}_t^{x,\phi} = d(e^{-rt}\Pi_t) = \phi_t d\tilde{S}_t$

$$\Rightarrow d\tilde{g}(S_T) = \partial_x V(t,S_t)d\tilde{S}_t$$

Q1. Define $P_t(x) := P(T=t, K=x)$, similar for call.

We have $P_2(90) = 6$. $P_2(110) = 14$

$C_1(100) = 11$

① By put-call parity:

$$C_1(100) + S_0 = P_1(100) + \underbrace{e^{-r\cdot 1}}_{=1} \cdot \underbrace{100}_{K.}$$

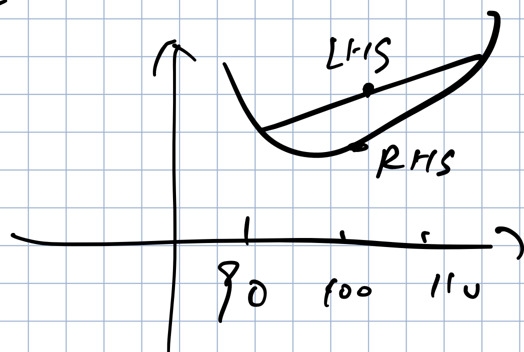
$$\Rightarrow P_1(100) = C_1(100) - \underbrace{S_0}_{=100} + 100 = \$11 \quad (\text{at initial time})$$

If we long a $P_1(100)$

②. By the convexity:

We should have

$$P_2(90) + P_2(110) \geq 2 \cdot P_2(100).$$



③. Recall: $P_2(100) \geq P_1(100)$

So we have:

$$P_2(90) + P_2(110) \geq 2 \cdot P_2(100)$$
$$\geq 2 \cdot P_1(100)$$

However. $P_2(90) + P_2(110) = 20.$, $2 \cdot P_1(100) = 2 \cdot C_1(100) = 22.$

But in reality: $P_2(90) + P_2(110) = 20 < 22 = 2 \cdot P_1(100)$

So either $P_2(90) + P_2(110)$ is lower than its actual value or $2P_1(100)$ is higher than its actual value.

Construct the portfolio: long $P_2(90)$, $P_2(110)$.

we short $\boxed{2\cdot P_1(100)}$

$$\Downarrow$$

We short $\boxed{2\cdot(C_1(100)-S_0+100)}$

$$\pi(t=0) = 6+14-2(11-\cancel{100}+\cancel{100})$$

$$= -2$$

$$\le 0$$

① $\pi(t=0)<0$, $\mathbb{P}[\pi(t=T)\ge 0]=1$, $\mathbb{P}[\pi(t=T)>0]>0$.

② $(=)$

$$\pi(t=1) = \underline{P_2(90, \text{at time } 1) + P_2(110, \text{at time } 1)}$$

$$-2[C_1(100, \text{at time } 1) - S_1 + 100]$$

$$\ge P_1(90, \text{at time } 1) + P_1(110, \text{at time } 1)$$

$$-2[(S_1-100)_+ - S_1 + 100]$$

$$= (90-S_1)_+ + (110-S_1)_+ - 2(S_1-100)_+ + 2(S_1-100)$$

$$= \begin{cases} 0, & S_1 > 110. \\ 110-S_1, & 100 < S_1 \le 110 \\ S_1-90, & 90 < S_1 \le 100 \\ 0, & S_1 \le 90 \end{cases} \ge 0$$

So $\mathbb{P}[\pi(t=1) \geq 0] = 1$, $\mathbb{P}[\pi(t=1) > 0] > 0$.
$\Rightarrow$ π is the arbitrage strategy

Midterm exam

Q3. Consider an option with maturity $T > 0$, with payoff :

$$g(x) = \begin{cases} x & \text{if } x \leq K_1 \\ \frac{K_1}{K_2 - K_1}(K_2 - x) & \text{if } K_1 < x \leq K_2 \\ x - K_2 & \text{if } x > K_2. \end{cases}$$

compute the option price $\mathbb{E}^{\mathbb{Q}}[e^{-rT} g(S_T)]$

$$\mathbb{E}^{\mathbb{Q}}[g(S_T)] = \mathbb{E}^{\mathbb{Q}}[\underbrace{S_T \mathbb{1}_{S_T \leq K_1}}_{②} + \frac{K_1}{K_2 - K_1}(K_2 - S_T)\mathbb{1}_{K_1 < S_T \leq K_2} + \underbrace{(S_T - K_2)\mathbb{1}_{S_T > K_2}}_{①}]$$

$e^{-rT} ① = e^{-rT}\mathbb{E}^{\mathbb{Q}}[(S_T - K)_+] =$ Call option price at time 0.

$$= S_0 \Phi(d_1(K_2)) - e^{-rT}\Phi(d_2(K_2))$$

where $d_1(K) = \frac{\log(S_0/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$, $d_2(K) = \frac{\log(S_0/K) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$

$e^{-rT} ② = e^{-rT}\mathbb{E}[S_T \mathbb{1}_{S_T \leq K_1}]$

For put options, $e^{-rT}\mathbb{E}^{\mathbb{Q}}[(K - S_T)_+]$

$$= e^{-rT}\mathbb{E}^{\mathbb{Q}}[(K - S_T)\mathbb{1}_{S_T \leq K}]$$

$$= e^{-rT}\mathbb{E}^{\mathbb{Q}}[K \mathbb{1}_{S_T \leq K_1}] - e^{-rT}\mathbb{E}^{\mathbb{Q}}[S_T \mathbb{1}_{S_T \leq K_1}]$$

$$e^{-rT} K \Phi(-d_2(K_1)) - (S_0 \Phi(-d_1(K_1)))$$

Call: $\underbrace{S_0\,\Phi(d_1)}_{\mathbb{E}[e^{-rT}S_T\mathbb{1}_{S_T\geq K}]} - \underbrace{e^{-rT}K\,\Phi(d_2)}_{\mathbb{E}[e^{-rT}K\mathbb{1}_{S_T\geq K}]}$

put: $e^{-rT}K\,\Phi(-d_2) - S_0\,\Phi(-d_1)$

$$\mathbb{E}^{\Theta}\left[(K_2 - S_T)\mathbb{1}_{K_1<S_T\leq K_2}\right]$$

$$= K_2\,\mathbb{E}^{\Theta}\left[\mathbb{1}_{K_1<S_T\leq K_2}\right] - \mathbb{E}^{\Theta}\left[S_T\,\mathbb{1}_{K_1<S_T\leq K_2}\right]$$

$$= K_2\,\Theta\left[K_1<S_T\leq K_2\right] - \left(\mathbb{E}^{\Theta}\left[S_T\left(\mathbb{1}_{S_T\leq K_2} - \mathbb{1}_{S_T\leq K_1}\right)\right]\right)$$

$$= K_2\left(\underbrace{\mathbb{E}^{\Theta}\left[\mathbb{1}_{S_T\leq K_2}\right]}_{\Phi(-d_2(K_2))} - \underbrace{\mathbb{E}^{\Theta}\left[S_T\leq K_1\right]}_{\Phi(-d_2(K_1))}\right) - \left(\underbrace{\mathbb{E}^{\Theta}\left[S_T\mathbb{1}_{S_T\leq K_2}\right]}_{\Phi(-d_1(K_2))} - \underbrace{\mathbb{E}^{\Theta}\left[S_T\mathbb{1}_{S_T\leq K_1}\right]}_{\Phi(-d_1(K_1))}\right)$$